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Finite-size analysis of two-dimensional $U(1)$ lattice gauge theories

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Abstract. We analyse the finite-size scaling of two-dimensional $U(1)$ lattice gauge theories with a θ-term interaction. The finite-size corrections to the specific heat, Binder–Landau and U_4 cumulants agree with the expected asymptotic behaviour for first-order phase transitions. However, we find that the leading correction to the position of the extremal points of these quantities is not universal. On the other hand, the finite-size corrections to the mass gap behave as for second-order phase transitions. In particular, the curves corresponding to different-size approximations do not cross in the vicinity of the transition points. The feature is associated to the existence of a divergent correlation length and holds for a wider class of models.

1. Introduction

In the last few years there has been renewed activity on the study of first-order phase transitions motivated in part by some controversies over the behaviour of the renormalization group in the vicinity of the transition and the criteria to determine the order of a transition from finite-size analysis. Although there are not many new results, two different rigorous approaches have shed some light on those problems.

For systems with bounded fluctuating variables and absolutely summable real Hamiltonians it has been rigorously proved that the renormalization group, when properly defined, is continuous and single-valued across the transition surface. The only real pathologies arise at some points where the renormalization group is not defined at all [1].

On the other hand, a rigorous theory of finite-size scaling has been developed for systems which admit a representation as contour models with small activities such as the Ising model at low temperature and the q-state Potts model with large enough q [2]. In such a case, for large lattices sizes $L \gg 1$ the partition function can be written as a sum of terms, each of them giving the contribution of small fluctuations around a pure phase, plus a remainder, which can be bounded by an exponentially decaying term $O(e^{-bL})$. Consequently, the asymptotic behaviour of the extremal points of the specific heat (C_v), Binder–Landau cumulant (U_{BL}) [3],

$$U_{BL} = \frac{1}{3}\left(1 - \frac{\langle E^4\rangle}{\langle E^2\rangle^2}\right) \quad (1)$$

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and U_4 ratio of cumulants,

$$U_4 = \frac{\langle (E - \langle E \rangle^4 \rangle}{\langle (E - \langle E \rangle)^2 \rangle^2} \tag{2}$$

in a d-dimensional system is given in terms of increasing powers of L^{-d} plus some exponentially decaying terms. If $Q(L)$ denotes the value of any of the quantities C_v/L^d, U_4 or $U_{\rm BL}$ in a square lattice of size L and $\beta_*^Q(L)$ the inverse of the temperatures of the corresponding extremal points, we have the following asymptotic expansions:

$$Q_*(L) = X_Q^{(0)} + X_Q^{(1)}/L^d + \mathcal{O}(L^{-2d}) \tag{3a}$$

$$\beta_*^Q(L) = \beta_c + Y_Q^{(1)}/L^d + Y_Q^{(2)}/L^{2d} + \mathcal{O}(L^{-3d}) \tag{3b}$$

where β_c is the inverse of the critical temperature. The precise values of the coefficients $\{X_Q^{(k)}\}_{k=0,1}$ $\{Y_Q^{(k)}\}_{k=1,2}$ are exactly known for the q-state Potts model for large values of q [4]. This behaviour has been verified numerically for the $q = 20$ Potts model in two-dimensional lattices [5]. However, in the case of lower q-state Potts models ($q = 7, 10$) Monte Carlo calculations for square lattices up to size $L = 50$ are not in agreement with this ansatz [4]. The disagreement might be due to the fact that the lattice size is not large enough to reach the asymptotic behaviour. Otherwise, it would mean that the finite-size effects of these models are not governed by the expansions (3a) and (3b).

In any case, the finite-size expansions (3a) and (3b) imply that the critical exponent $\nu = 1/d$ is in agreement with the value predicted by the discontinuity fixed-point scenario of the renormalization group behaviour for first-order transitions [6]. Moreover, if $X^{(0)}_{C_v/L^d} \neq 0$, the finite-size analysis becomes similar to that of second-order phase transitions with $\alpha = 1$ and only the existence of as many critical exponents $y = d$ as different phases which can coexist is the main signal of the first-order transitions [6].

In this note we analyse these problems in a two-dimensional Abelian gauge model with a θ-term interaction on a lattice. The model is equivalent to the $q \to \infty$ limit of a family of one-dimensional clock models introduced by two of us [7]. The simplicity of the model implies that it can be solved exactly and the quantities can be computed exactly without numerical errors.

2. Finite-size scaling

The partition function of the $\mathbb{Z}_q$ gauge model in a two-dimensional square lattice of size $N \times N$ with periodic boundary conditions is given by

$$\mathcal{Z}(J, \epsilon, L) = \mathrm{Tr}\ T^{N^2} \tag{4}$$

where

$$T_{pp'} = \exp\left\{-J\cos\frac{2\pi}{q}(p-p') - \mathrm{i}\epsilon\sin\frac{2\pi}{q}(p-p')\right\} \qquad p, p' = 0, 1, \ldots, q-1 \tag{5}$$

is the transfer matrix, $L = N^2$ the number of plaquetes of the lattice, and J and ϵ the real coupling constants associated, respectively, to the gauge coupling and the θ-term interaction in Wilson's action [8, 9],

$$S = J\sum_P \mathrm{tr}\ \mathrm{Re}(U_P) + \mathrm{i}\epsilon\sum_P \mathrm{tr}\ \mathrm{Im}(U_P). \tag{6}$$

The model can also be seen as a one-dimensional q-state clock model [7] and as a chiral q-state Potts model. Due to the presence of the imaginary terms in the transfer matrix (5), the model undergoes first-order phase transitions [7].

The behaviour of the renormalization group for $q = 3$ and $q = 4$ agrees with the discontinuity fixed-point scenario and presents pathologies similar to the Griffiths–Pearce pathologies [10]. Although the models are very peculiar because of the complex character of the Hamiltonian, some of our conclusions are also relevant for models with real-valued interactions with first-order transitions of infinite correlation length.

In this note we will concentrate on the analysis of the behaviour of finite-size corrections to the $U(1)$ lattice gauge theory, which corresponds to the $q \to \infty$ limit. The eigenvalues of the transfer matrix are [7]

$$\lambda_k = \begin{cases} 2\pi \left(\dfrac{J+\epsilon}{J-\epsilon}\right)^{k/2} I_{|k|}(\sqrt{J^2-\epsilon^2}) & |\epsilon| \leqslant J \\ 2\pi \left(\dfrac{J+\epsilon}{\epsilon-J}\right)^{k/2} J_{|k|}(\sqrt{\epsilon^2-J^2}) & |\epsilon| > J \end{cases} \tag{7}$$

where I_k and J_k are the Bessel functions of integer order. If we consider periodic boundary conditions, only the leading eigenvalue of the transfer matrix is relevant for the thermodynamic limit. The phase transitions occur at the points (J, ϵ) of the parameter space where the leading eigenvalue is degenerate. The energy density is discontinuous at these points, thus the transitions are first order, although they have a divergent correlation length $\xi = \infty$. This unusual feature is due to the complex character of the action. Furthermore, the critical index ν associated to the divergency of the correlation length is $\nu = 1$, in agreement with the discontinuity fixed-point picture for first-order phase transitions [6].

For simplicity we will only consider phase-transition points with $J = \epsilon$, but the results can easily be generalized for all transition points. In this case the eigenvalues of the transfer matrix (7) read

$$\lambda_k = \begin{cases} 2\pi J^k/k! & k \geqslant 0 \\ 0 & k < 0 \end{cases} \tag{8}$$

which implies the existence of first-order phase transitions at the points

$$J_c^{(k)} = k+1 \qquad k = 0, 1, 2, \ldots . \tag{9}$$

The mean energy density is given by

$$u(J, L) = \langle E \rangle_L = \frac{1}{L}\frac{\partial}{\partial J} \log Z(J, L) \tag{10}$$

where

$$Z(J, L) = \sum_{\{p_n\}} \mathrm{e}^{-EL} = \sum_{k \geqslant 0} (\lambda_k)^L \tag{11}$$

is the partition function of a periodic chain of length L. In the thermodynamic limit ($L \to \infty$) we recover the internal energy density $u(J) = \langle E \rangle = \lim_{L\to\infty} \langle E \rangle_L$.

The correlation function of two parallel loops Γ_1 Γ_2 wrapping the periodic lattice in opposite directions at a distance m (see figure 1),

$$\langle U_{\Gamma_1} U_{\Gamma_2} \rangle_L = \frac{1}{2} \frac{\sum_{k\geqslant 0} \left(\frac{J^k}{k!}\right)^L \left\{ \left(\frac{J}{k+1}\right)^m + \left(\frac{k}{J}\right)^m \right\}}{\sum_{k\geqslant 0} \left(\frac{J^k}{k!}\right)^L} \tag{12}$$

is dominated in that limit ($m \ll L$) by the leading eigenvalue [11]

$$\langle U_{\Gamma_1} U_{\Gamma_2} \rangle = \lim_{L\to\infty} \langle U_{\Gamma_1} U_{\Gamma_2} \rangle_L = \frac{1}{2} \left\{ \left(\frac{J}{1+[J]}\right)^m + \left(\frac{[J]}{J}\right)^m \right\} \tag{13}$$

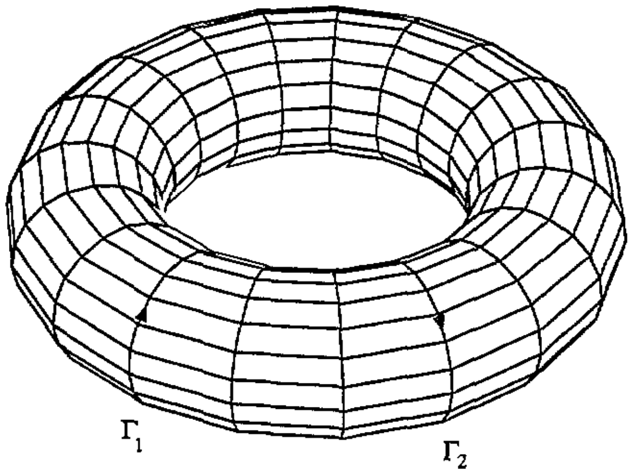

Figure 1. Two parallel Wilson loops Γ_1 Γ_2 wrapping the periodic lattice in opposite directions.

where $[J]$ denotes the integer part of J. Near the transition point the correlation function $\langle U_{\Gamma_1} U_{\Gamma_2}\rangle$ has an asymptotic exponentially decaying behaviour

$$\langle U_{\Gamma_1} \cdot U_{\Gamma_2}\rangle \sim A\,\mathrm{e}^{-\mu(J)m} \tag{14}$$

as m goes to ∞, with mass gap $\mu(J)$ given by

$$\mu(J) = \frac{|J - J_c^{(k)}|}{J_c^{(k)}} \tag{15}$$

for $|J - J_c^{(k)}| \ll J_c^{(k)}$. Thus, the correlation length $\xi(J) = \mu(J)^{-1}$ diverges as J approaches the critical point with a critical exponent $\nu = 1$.

On the other hand, one can expand (13) around a transition point. Using the analogue of (14) for finite L we get an expansion for the mass gap in powers of $\eta = J - J_c^{(k)}$ and N plus some negligible exponential terms. In the particular case $J_c^{(0)} = 1$ the result is

$$\mu^2(J = 1 + \eta, L) = 2^{-L/2}\log^2 2 + \eta\mathrm{e}^{-L/2}\left[L\log^2 2 - 2\log 2\right] \\ +\eta^2\left[1 + \mathcal{O}(2^{-L/2})\right] + \mathcal{O}(\eta^3, 2^{-L}). \tag{16}$$

The minimum value of the mass gap

$$\mu^2_{\min} = 2^{-L/2}\log^2 2 + \mathcal{O}(2^{-L}) \tag{17}$$

is reached at the points

$$J(\mu^2_{\min}) = 1 - \frac{L}{2}\mathrm{e}^{-L/2}\log^2 2\left(1 - \frac{2}{L\log 2}\right) + \mathcal{O}(2^{-L}) \tag{18}$$

which is in agreement with the results obtained above for the thermodynamic limit $L \to \infty$. In figure 2 we have plotted the function $\mu(J, L)$ near $J_c^{(0)}$ for several values of L.

An interesting property of the model is that the derivative of the mass gap with respect *to L is negative near a transition point. In particular, in a neighborhood of $J_c^{(0)}$ we have*

$$\frac{\mathrm{d}\mu^2}{\mathrm{d}L} = -\frac{L}{2}2^{-L/2}\log^2 2\left\{\log 2 + \mathcal{O}(\eta)\frac{1}{L}\right\} < 0. \tag{19}$$

This means that the curves $\mu(J, L)$ do not intersect each other near the transition point (see figure 2). This behaviour does not agrees with the picture advocated in [12] to discriminate

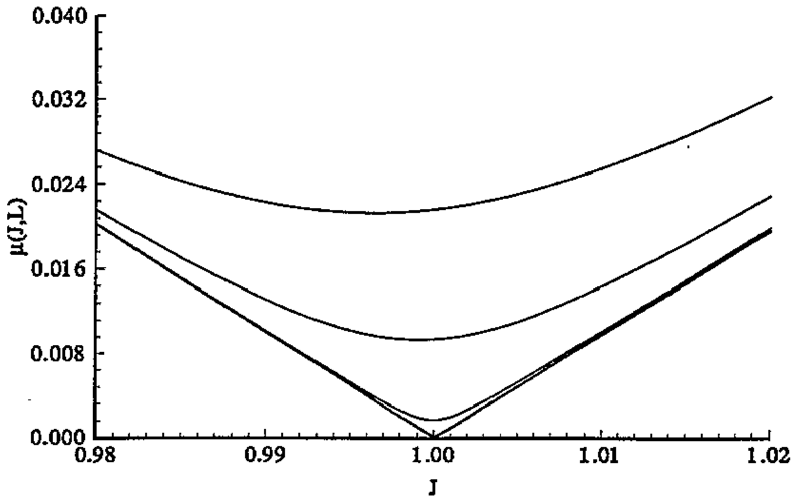

Figure 2. Dependence of the mass gap $\mu(J, L)$ on the coupling J in a neighbourhood of $J_c^{(0)} = 1$ for different values of the chain length: $L = 20$, 25 and 30. For $L \geqslant 40$ the curves overlap almost perfectly with the thermodynamic result (thicker curve).

between first- and second-order phase transition. In this model the curves $\mu(J, L)$ which describe the finite-size approximation to the mass gap do not cross each other in the vicinity of first-order phase-transition points. Thus, the absence of crossing of these curves cannot be taken as a signal of second-order phase transition for any statistical-mechanical model. In fact, it is likely to be associated to the infinite correlation length of the system irrespective of the first- or second-order nature of the phase transition. In particular, it can be shown that the same behaviour arises for all spin models whose lowest energy levels are degenerate with the only condition that the spin variables must have a non-vanishing correlation between some of those degenerate eigenstates.

In the thermodynamic limit the specific heat is a non-positive and discontinuous function of the coupling constant J

$$C_v(J) = -[J]. \tag{20}$$

The jumps on the specific heat are associated to the different phase transitions.

Near the phase-transition points the finite-size corrections

$$C_v(J, L) = J^2 L \left(\langle E^2 \rangle_L - \langle E \rangle_L^2 \right) \tag{21}$$

can be expanded in terms of powers in $\eta = J - J_c^{(k)}$, $1/L$ and exponentially decaying terms of the form $\exp(-aL)$. If we neglect the exponential terms, $C_v(J_c^{(k)} + \eta, L)$ reduces to a polynomial in η with L-dependent coefficients. In this way we obtain the coupling $J(C_v^{\max})$ where the specific heat C_v reaches a maximum value $C_v^{\max}$. The results for $J_c^{(0)} = 1$ are given by

$$J(C_v^{\max}) = 1 - \frac{2}{L^2} + \mathcal{O}(L^{-4}) \tag{22a}$$

$$C_v^{\max} = \frac{L}{4} - \frac{1}{2} + \frac{1}{4L} + \mathcal{O}(2^{-L}). \tag{22b}$$

The asymptotic behaviour of those quantities is similar to that obtained for the Potts model. However, we find a clear difference in $J(C_v^{\max})$. The leading correction to $J(C_v^{\max})$

is $\mathcal{O}(L^{-2})$, instead of $\mathcal{O}(L^{-1})$. In the discontinuity fixed-point interpretation this means that the shift $J(C_v^{\max}) - J_c$ does not scale as $L^{-1/\nu}$.

Nevertheless, the specific heat behaves as predicted by the Borgs–Kotecký theory, which is in agreement with a critical exponent $\alpha = 1$. Finally, we note that the value of C_v at the transition point picks up only exponentially small corrections, as observed in [4]

$$C_v(1, L) = \tfrac{1}{4}L - \tfrac{1}{2} + \mathcal{O}(2^{-L}). \tag{23}$$

This means that the asymptotic regime is reached much faster at the thermodynamic transition point $J = 1$ than at the finite-size maximal point $J(C_v^{\max})$ (see figure 3).

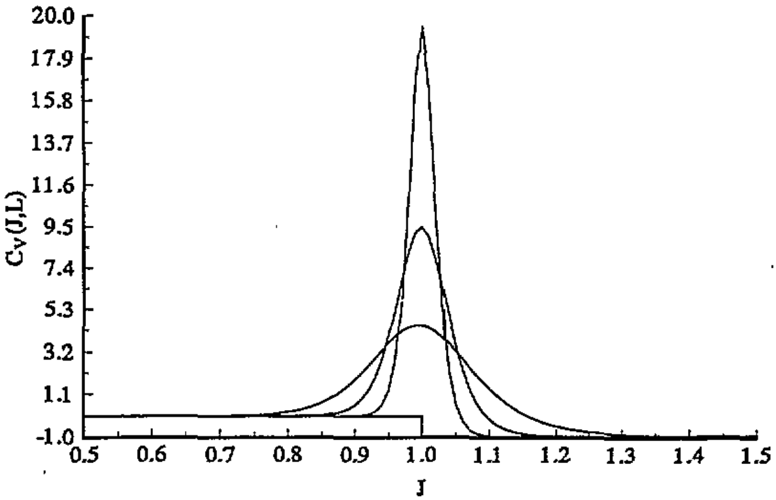

Figure 3. Behaviour of the specific heat function $C_v(J, L)$ in the same region as in figure 2. We plot the curves for $L = 20, 40$ and 80 together with the thermodynamic limit (thicker curve).

Now we examine the behaviour of the $U_4(J)$ ratio of cumulants in a finite chain,

$$U_4(J, L) = \frac{\langle (E - \langle E \rangle_L)^4 \rangle_L}{\langle (E - \langle E \rangle_L)^2 \rangle_L^2}. \tag{24}$$

In the thermodynamic limit it takes the following form:

$$U_4(J) = \begin{cases} \infty & J < 1 \\ 3 & J > 1 \qquad J \neq J_c^{(k)} \\ 1 & J = J_c^{(k)} \quad k = 0, 1, 2, \ldots . \end{cases} \tag{25}$$

The divergency of $U_4(J)$ for $J \in [0, 1)$ is due to the fact that the density energy is constant in this interval. Notice that the value $U_4(J_c^{(k)}) = 1$ agrees with the expected result for first-order transitions.

Repeating the analysis of finite-size approximation for the U_4 ratio of cumulants, we get the following values for its minimum value:

$$U_4^{\min} = 1 - \frac{8}{L} - \frac{4}{L^2} + \mathcal{O}(L^{-3}) \tag{26}$$

and its position

$$J(U_4^{\min}) = 1 - \frac{4}{L^2} + \mathcal{O}(L^{-3}) \tag{27}$$

in the vicinity of the transition point $J = J_c^{(0)}$. Notice that $U_4 < 1$ and this property can never arise for systems with real-valued interactions, but in this case it is generated by the complex character of the action. The leading correction to the position of the minimum is again of order $\mathcal{O}(L^{-2})$.

We now repeat the same analysis for the Binder–Landau cumulant,

$$U_{\mathrm{BL}}(J, L) = \frac{1}{3}\left(1 - \frac{\langle E^4\rangle_L}{\langle E^2\rangle_L^2}\right) \tag{28}$$

which in the thermodynamic limit takes the form

$$U_{\mathrm{BL}}(J) = \begin{cases} -\infty & J < 1 \\ 0 & J > 1 \quad J \neq J_c^{(k)} \\ -\dfrac{1}{12}\dfrac{(1+2k)^2}{k^2(1+k)^2} & J = J_c^{(k)} \quad k = 0, 1, 2, \ldots . \end{cases} \tag{29}$$

The explicit value for $U_{\mathrm{BL}}(J_c^{(k)})$ can be obtained following [13] and using the fact that the internal energy at the transition points jumps from $k/(k+1)$ to 1.

The maximum value of the cumulant and its position near $J_c^{(1)} = 2$ are given by (see figure 4)

$$U_{\mathrm{BL}}^{\min} = -\frac{3}{16} - \frac{35}{32L} - \frac{751}{1728L^2} + \mathcal{O}(L^{-3}) \tag{30a}$$

$$J(U_{\mathrm{BL}}^{\min}) = 2 - \frac{2\log 4}{L} + \frac{23 + 9\log^2 4}{9L^2} + \mathcal{O}(L^{-3}). \tag{30b}$$

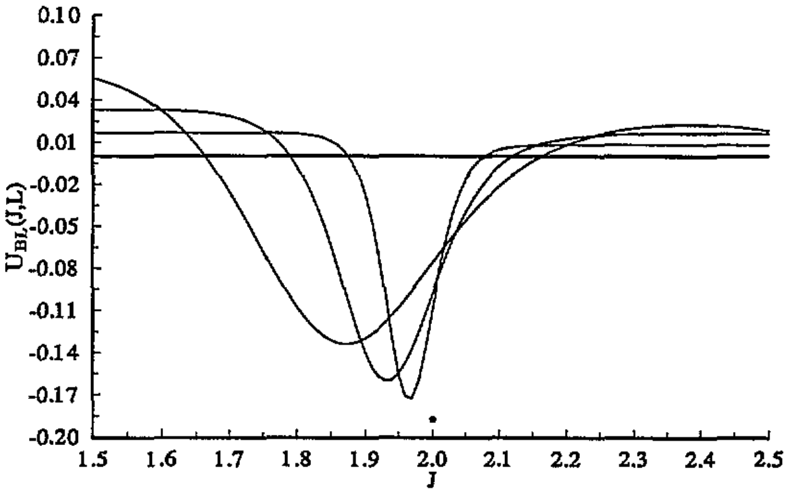

Figure 4. Finite-size behaviour of the Binder–Landau cumulant $U_{\mathrm{BL}}(J, L)$ near $J_c^{(1)} = 2$. The symbol • marks the thermodynamic limit at $J_c^{(1)} = 2$.

The leading correction to $J(U_{\mathrm{BL}}^{\min})$ is, amazingly, of order $\mathcal{O}(L^{-1})$, which is in full agreement with the Borgs–Kotecký theory, but slightly different to the results for the other cumulants. In this model only the Binder–Landau cumulant scales as $L^{-1/\nu}$.

3. Conclusions

In summary, the values of $C_v^{\max}/L^d$, $U_{\rm BL}^{\min}$ and $U_4^{\min}$ are in agreement with those associated to first-order transition points and their finite-size corrections also behave as predicted for such types of transitions. The complex character of the action only generates some minor modifications: the specific heat is non-positive and the U_4 ratio of cumulants is smaller than one for some values of J.

The scaling of the position of the minimum of $U_{\rm BL}(J, L)$ is also in agreement with the theoretical predictions: the leading correction behaves as $L^{-1/\nu} = L^{-d}$. However, the leáding corrections for the extremal points of $U_4(J, L)$ and $C_v(J, L)$ are of order L^{-2}. This behaviour does not impliy the failure of the standard finite-size scaling [14], but simply means that the leading correction is not *universal*, because it might depend on the quantity considered.

Finally, the finite-size approximations to the mass gap in this model do not behave as for the first-order phase transitions of Ising and Potts models in two dimensions. The mass gap curves $\mu(J, L)$ obtained for different sizes of the system do not cross each other in the vicinity of first-order transition points. Therefore this property cannot be used as an indicator of the order of the phase transition [12]. It indicates rather the existence of an infinity-length correlation in the system.

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